

DIRECT SUMS OF ZERO PRODUCT DETERMINED ALGEBRAS

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ABSTRACT. We reformulate the definition of a zero product determined algebra in terms of tensor products and obtain necessary and sufficient conditions for an algebra to be zero product determined. These conditions allow us to prove that the direct sum $\bigoplus_{i \in I} A_i$ of algebras for any index set I is zero product determined if and only if each of the component algebras A_i is zero product determined. As an application, every parabolic subalgebra of a finite-dimensional reductive Lie algebra, over an algebraically-closed field of characteristic zero, is zero product determined. In particular, every such reductive Lie algebra is zero product determined.

Let \mathbb{K} be a commutative ring. Given a \mathbb{K} -algebra A and a \mathbb{K} -bilinear map $\varphi : A \times A \longrightarrow B$, we may ask whether or not φ may be written as the composition of multiplication in A with a \mathbb{K} -linear transformation $\tilde{\varphi}$, that is, whether or not

$$\varphi(a_1, a_2) = \tilde{\varphi}(a_1 a_2)$$

for all $a_1, a_2 \in A$ for some $\tilde{\varphi} : A^2 \longrightarrow B$. (Here and throughout, A^2 denotes the \mathbb{K} -linear span of the products of members of A).

In order to study the above problem, Brešar, Grašič, and Ortega introduced the notion of a zero product determined algebra in [5]. A \mathbb{K} -algebra A (not necessarily associative) is called *zero product determined* if each \mathbb{K} -bilinear map $\varphi : A \times A \longrightarrow B$ satisfying

$$\varphi(a_1, a_2) = 0 \text{ whenever } a_1 a_2 = 0$$

can be written as $\varphi(a_1, a_2) = \tilde{\varphi}(a_1 a_2)$ for some $\tilde{\varphi} : A^2 \longrightarrow B$. Their definition was motivated by applications to the study of zero product preserving linear maps defined on Banach algebras and on matrix algebras under the standard matrix product, the Lie product, or the Jordan product [1, 5, 7, 8, 13]. Let A, B be \mathbb{K} -algebras, and let $f : A \longrightarrow B$ be \mathbb{K} -linear. f is said to be zero product preserving if $f(a_1)f(a_2) = 0$

whenever $a_1a_2 = 0$. One would like to find conditions on f or on A that imply that f is a homomorphism of \mathbb{K} -algebras, or is at least close to an algebra homomorphism in the following sense. We define a mapping $\varphi(a_1, a_2) = f(a_1)f(a_2)$. Then φ is \mathbb{K} -bilinear and satisfies $\varphi(a_1, a_2) = 0$ whenever $a_1a_2 = 0$. If A is zero product determined, then there is a unique \mathbb{K} -linear $\tilde{\varphi} : A^2 \rightarrow B$ satisfying

$$\tilde{\varphi}(a_1a_2) = \varphi(a_1, a_2) = f(a_1)f(a_2)$$

for all $a_1, a_2 \in A$. If we further assume that $1_A \in A$ (so in particular $A^2 = A$), then

$$\tilde{\varphi}(a) = \tilde{\varphi}(1_Aa) = \varphi(1_A, a) = f(1_A)f(a)$$

for all $a \in A$, and by combining the above equations, we arrive at

$$f(a_1)f(a_2) = f(1_A)f(a_1a_2)$$

for all $a_1, a_2 \in A$. As a corollary, if A is zero product determined, and if $1_A \in A$ and $1_B \in B$ are identities, then any zero product preserving linear map $f : A \rightarrow B$ that satisfies $f(1_A) = f(1_B)$ is an algebra homomorphism.

The initial work of Brešar, Grašič, and Ortega and subsequent work by Ge, Grašič, Li, and Wang have provided examples both of zero product determined algebras and algebras that are not zero product determined [9, 13]. Recent similar work includes Brešar's and Šmerl's study of commutativity preserving linear map [6] and Chen's, Wang's, and Yu's study of idempotent preserving bilinear maps and the notion of an idempotent elements determined algebra [12].

In the context of Lie algebras, Grašič in [9] gives an example of an infinite-dimensional Lie algebra that is not zero product determined. Brešar, Grašič, and Ortega produce a family of Lie algebras of arbitrarily-large finite dimension that are not zero product determined in [5]. In [9], Grašič shows that the classical Lie algebras over an arbitrary field are zero product determined. Chen, Wang, and Yu compliment this result in [13] by showing that every parabolic subalgebra of a finite-dimensional simple Lie algebra over an algebraically-closed field of characteristic zero is zero product determined. In particular, the simple Lie algebras themselves are zero product determined, assuming the scalar field to be algebraically closed and of characteristic zero.

In this paper, we provide three main results that further research in this direction. We reformulate the definition of a zero product determined algebra in terms of tensor products and obtain necessary and

sufficient conditions for an algebra to be zero product determined (Theorem 2.4). These necessary and sufficient conditions allow us to prove that the direct sum of algebras $\bigoplus_{i \in I} A_i$ for any index set I is zero product determined if and only if each component algebras A_i is zero product determined (Theorem 3.3). We then apply this result in the context of Lie algebras, showing that all finite-dimensional reductive Lie algebras over an algebraically-closed field of characteristic zero, and all parabolic subalgebras of such Lie algebras, are zero product determined (Theorem 4.2).

In all that follows, let \mathbb{K} denote a fixed commutative ring. By a *module* we mean a \mathbb{K} -module. The notation $S \leq A$ means S is a submodule of A . If A is a module and $S \subset A$, let $\langle S \rangle$ denote the submodule of A generated by S .

By a *linear map*, we mean a \mathbb{K} -linear map between two modules. By a *bilinear map*, we mean a \mathbb{K} -bilinear map between the Cartesian product of two modules and a third module. We say φ *factors through* ψ to mean that $\varphi = \tilde{\varphi}\psi$ for some linear map $\tilde{\varphi}$.

1. PRELIMINARIES

We will often make use of the following classical result in module theory.

Lemma 1.1. *Let A, B, C be modules. Let $\varphi : A \rightarrow B$ be a linear map, and let $\rho : A \rightarrow C$ be a surjective linear map. There exists a linear map $\tilde{\varphi} : C \rightarrow B$ satisfying $\varphi = \tilde{\varphi}\rho$ if and only if $\text{Ker } \rho \subset \text{Ker } \varphi$, and in such a case, the map $\tilde{\varphi}$ is uniquely determined.*

$$\begin{array}{ccc} A & \xrightarrow{\rho} & C \\ & \searrow \varphi & \swarrow \tilde{\varphi} \\ & B & \end{array}$$

Proof. For sufficiency, one is referred to the proof given of Theorem 7 in Chapter VI, Section 4 of [3].

For necessity, simply note that if such a $\tilde{\varphi}$ exists, and if $a \in \text{Ker } \rho$, then

$$\varphi(a) = \tilde{\varphi}\rho(a) = \tilde{\varphi}(0) = 0.$$

□

The *tensor product* of two modules A and B is denoted by

$$A \otimes B = A \otimes_{\mathbb{K}} B = \mathcal{F}(A \times B)/S$$

where $\mathcal{F}(A \times B)$ is the free module taking for generating set the Cartesian product $A \times B$, and S is the submodule generated by elements of the type $(a_1 + ka_2, b) - (a_1, b) - k(a_2, b)$ and $(a, b_1 + kb_2) - (a, b_1) - k(a, b_2)$ for all $k \in \mathbb{K}$, $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$. We write $a \otimes b$ to denote the coset $(a, b) + S$, and we call elements of this type *pure tensors*. Let

$$T = \{a \otimes b \mid a \in A, b \in B\}$$

denoted the set of all pure tensors. Every element of $A \otimes B$ is a sum of finitely-many pure tensors.

Lemma 1.1 implies the following result.

Lemma 1.2. *Let A, B, C be modules. There is a one-to-one correspondence between the set of bilinear maps $\varphi : A \times B \longrightarrow C$ and the set of linear maps $\bar{\varphi} : A \otimes B \longrightarrow C$, given by*

$$\bar{\varphi} \left(\sum_i a_i \otimes b_i \right) = \sum_i \varphi(a_i, b_i) \quad \text{and} \quad \varphi(a, b) = \bar{\varphi}(a \otimes b)$$

for all $a, a_i \in A$ and $b, b_i \in B$.

Proof. Suppose $\bar{\varphi} : A \otimes B \longrightarrow C$ is a linear map. Let $P(\bar{\varphi}) : A \times B \longrightarrow C$ be a map defined by $P(\bar{\varphi})(a, b) = \bar{\varphi}(a \otimes b)$ for all $a \in A$ and $b \in B$. Obviously $P(\bar{\varphi})$ is a well-defined bilinear map.

Conversely, suppose $\varphi : A \times B \longrightarrow C$ is a bilinear map. Since $\mathcal{F}(A \times B)$ is the free module on $A \times B$, the map φ induces a linear map $\mathcal{F}(\varphi) : \mathcal{F}(A \times B) \longrightarrow C$, such that

$$\mathcal{F}(\varphi) \left(\sum_i k_i (a_i, b_i) \right) = \sum_i k_i \varphi(a_i, b_i), \quad k_i \in \mathbb{K}.$$

The canonical projection $\pi : \mathcal{F}(A \times B) \longrightarrow \mathcal{F}(A \times B)/S = A \otimes B$ is a linear map with $\text{Ker } \pi = S$. We claim that $S \subset \text{Ker } \mathcal{F}(\varphi)$. $(a_1 + ka_2, b) - (a_1, b) - k(a_2, b)$ is a typical generator of S , and

$$\begin{aligned} & \mathcal{F}(\varphi) ((a_1 + ka_2, b) - (a_1, b) - k(a_2, b)) \\ &= \varphi(a_1 + ka_2, b) - \varphi(a_1, b) - k\varphi(a_2, b) = 0. \end{aligned}$$

The derivation is similar for the other typical generator $(a, b_1 + kb_2) - (a, b_1) - k(a, b_2)$, verifying the claim.

By Lemma 1.1, there exists a linear map $Q(\varphi) : A \otimes B \longrightarrow C$ such that $\mathcal{F}(\varphi) = Q(\varphi)\pi$. We see that

$$\begin{aligned} Q(\varphi) \left(\sum_i a_i \otimes b_i \right) &= Q(\varphi)\pi \left(\sum_i (a_i, b_i) \right) \\ &= \mathcal{F}(\varphi) \left(\sum_i (a_i, b_i) \right) = \sum_i \varphi(a_i, b_i) \end{aligned}$$

for all $a_i \in A$ and $b_i \in B$.

To prove the one-to-one correspondence in the statement, it remains to show that the functors $\bar{\varphi} \longmapsto P(\bar{\varphi})$ and $\varphi \longmapsto Q(\varphi)$ are inverse to each other. On one hand, given the linear map $\bar{\varphi} : A \otimes B \longrightarrow C$, we have

$$\begin{aligned} Q(P(\bar{\varphi})) \left(\sum_i a_i \otimes b_i \right) &= \sum_i P(\bar{\varphi})(a_i, b_i) \\ &= \sum_i \bar{\varphi}(a_i \otimes b_i) = \bar{\varphi} \left(\sum_i a_i \otimes b_i \right) \end{aligned}$$

and thus $Q(P(\bar{\varphi})) = \bar{\varphi}$; on the other hand, given the bilinear map $\varphi : A \times B \longrightarrow C$, we have

$$P(Q(\varphi))(a, b) = Q(\varphi)(a \otimes b) = \varphi(a, b)$$

and thus $P(Q(\varphi)) = \varphi$. This completes the proof. \square

2. ZERO PRODUCT DETERMINED ALGEBRAS

Definition 2.1. An *algebra* (more precisely, a \mathbb{K} -algebra) is a pair (A, μ) where A is a module and $\mu : A \otimes A \longrightarrow A$ is a linear map.

This definition encompasses associative algebras, alternative algebras (an example being the octonions), Leibniz algebras, Lie algebras, and Jordan algebras, among others. The algebra multiplication is encoded by the map μ if we define $a_1 a_2 = \mu(a_1 \otimes a_2)$. In light of Lemma 1.2, the condition $\mu : A \otimes A \longrightarrow A$ is equivalent to the requirement that multiplication be bilinear. We will denote by A^2 the submodule $\text{Im } \mu \leq A$. Notice that A^2 consists of finite sums of products in A , not merely the products themselves.

Definition 2.2. The algebra (A, μ) is called *zero product determined* if whenever a linear map $\varphi : A \otimes A \longrightarrow B$ satisfies

$$\mu(a_1 \otimes a_2) = 0 \text{ implies } \varphi(a_1 \otimes a_2) = 0,$$

then φ factors through μ :

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A^2 \\ & \searrow \varphi & \swarrow \tilde{\varphi} \\ & B & \end{array}$$

By Lemma 1.2, this definition is in agreement with that given in [5].

We will reformulate this definition in terms of kernels and pure tensors below. First, we require some terminology.

Consider an algebra (A, μ) , and recall that $T = \{a_1 \otimes a_2 | a_1, a_2 \in A\}$ denotes the set of pure tensors. For any map $\psi : A \otimes A \longrightarrow B$, let

$$T_\psi = T \cap \text{Ker } \psi.$$

In other words, T_ψ is the set comprised of those pure tensors whose image under ψ is 0. In particular, T_μ is the set of pure tensors of members of A whose product in A is 0.

Lemma 2.3. (A, μ) is zero product determined if and only if for every linear map $\varphi : A \otimes A \longrightarrow B$, we have

$$T_\mu \subset T_\varphi \text{ implies that } \text{Ker } \mu \subset \text{Ker } \varphi.$$

Proof. The two conditions " $T_\mu \subset T_\varphi$ " and " $\mu(a_1 \otimes a_2) = 0$ implies $\varphi(a_1 \otimes a_2) = 0$ " are literally identical, and " $\text{Ker } \mu \subset \text{Ker } \varphi$ " is equivalent to φ factoring through μ by Lemma 1.1. \square

Our first main result, stated below, gives a very clear conception of what makes an algebra (A, μ) zero product determined. First, recall that $\langle T_\mu \rangle$ denotes the module generated by T_μ .

Theorem 2.4. (A, μ) is zero product determined if and only if

$$\langle T_\mu \rangle = \text{Ker } \mu.$$

Proof. If $\langle T_\mu \rangle = \text{Ker } \mu$, then $T_\mu \subset T_\varphi$ implies that $\text{Ker } \mu = \langle T_\mu \rangle \leq \langle T_\varphi \rangle \leq \text{Ker } \varphi$, so φ factors through μ whenever $T_\mu \subset T_\varphi$. On the other hand, if $\langle T_\mu \rangle \neq \text{Ker } \mu$, the canonical projection $\pi : A \otimes A \longrightarrow$

$(A \otimes A)/\langle T_\mu \rangle$ satisfies $T_\mu \subset T_\pi$ but by Lemma 1.1, π does not factor through μ , so (A, μ) is not zero product determined. \square

The condition $\langle T_\mu \rangle = \text{Ker } \mu$ is just to say that $\text{Ker } \mu$ is generated by its intersection with T , i.e. by pure tensors. Thus, the theorem says that an algebra A is zero product determined if and only if the kernel of its multiplication map is generated by pure tensors. It is worthwhile to observe that while $A \otimes A$ is generated by T , an arbitrary submodule $B \leq A \otimes A$ is not necessarily generated by its intersection with T . We give a few examples to illustrate this point.

Proposition 2.5. *Suppose \mathbb{K} is a field. Let V be a vector space over \mathbb{K} with $\dim V \geq 2$. Then, neither the tensor algebra $\mathcal{T}(V)$ nor the symmetric algebra $\mathcal{S}(V)$ are zero product determined.*

Proof. Recall that the tensor algebra $\mathcal{T}(V)$ over a vector space V may be thought of as the free associative \mathbb{K} -algebra on $\dim V$ generators. Likewise, the symmetric algebra $\mathcal{S}(V)$ over V may be thought of as the free commutative associative \mathbb{K} -algebra on $\dim V$ generators.

The tensor and symmetric algebras over a vector space are integral domains, and as such, we have that $\mu(t_1 \otimes t_2) = 0$ if and only if $t_1 = 0$ or $t_2 = 0$, in either case giving $t_1 \otimes t_2 = 0$. In short, this means $T_\mu = 0$. To show that these algebras are not zero product determined, we must now show that $\text{Ker } \mu \neq 0$. In other words, we must produce at least one non-trivial kernel element.

Since $\dim V \geq 2$, we may select two \mathbb{K} -linearly independent vectors $v_1, v_2 \in V$.

Then for the tensor algebra $\mathcal{T}(V)$, consider the element

$$v_1 v_2 \otimes v_1 - v_1 \otimes v_2 v_1 \in \mathcal{T}(V) \otimes \mathcal{T}(V).$$

(Here, multiplication in $\mathcal{T}(V)$ is denoted by juxtaposition, to avoid confusion with members of $\mathcal{T}(V) \otimes \mathcal{T}(V)$.) We have that

$$\mu(v_1 v_2 \otimes v_1 - v_1 \otimes v_2 v_1) = v_1 v_2 v_1 - v_1 v_2 v_1 = 0$$

so that $v_1 v_2 \otimes v_1 - v_1 \otimes v_2 v_1 \in \text{Ker } \mu$, while also $v_1 v_2 \otimes v_1 - v_1 \otimes v_2 v_1 \neq 0$ by linear independence.

As for the symmetric algebra $\mathcal{S}(V)$, we may use the element

$$v_1 \otimes v_2 - v_2 \otimes v_1 \in \mathcal{S}(V) \otimes \mathcal{S}(V)$$

which is non-zero yet contained in $\text{Ker } \mu$ by commutativity of $\mathcal{S}(V)$. \square

The next proposition shows that the condition on the dimension of V in the previous is quite necessary. We recall that \mathbb{K} is an arbitrary commutative ring. When viewed as a module, \mathbb{K} has an obvious algebra structure by letting $\mu(k_1 \otimes k_2) = k_1 k_2$. Furthermore, any ideal H of \mathbb{K} is a submodule of \mathbb{K} , and the quotient module \mathbb{K}/H , again through the usual multiplication, has an algebra structure.

Proposition 2.6. *Assume $1 \in \mathbb{K}$. For any ideal $H \leq \mathbb{K}$, the ring \mathbb{K}/H is zero product determined when viewed as a \mathbb{K} -algebra. In particular, \mathbb{K} is zero product determined as a \mathbb{K} -algebra.*

Proof. $\mathbb{K}/H \otimes_{\mathbb{K}} \mathbb{K}/H \cong \mathbb{K}/H$, with μ serving as an isomorphism. This is because

$$\begin{aligned} (k_1 + H) \otimes_{\mathbb{K}} (k_2 + H) &= k_1(1 + H) \otimes_{\mathbb{K}} (k_2 + H) \\ &= (1 + H) \otimes_{\mathbb{K}} k_1(k_2 + H) \\ &= (1 + H) \otimes_{\mathbb{K}} (k_1 k_2 + H), \end{aligned}$$

so $k_1 k_2 + H = 0 + H$ (which is to say $\mu((k_1 + H) \otimes_{\mathbb{K}} (k_2 + H)) = 0$) implies that

$$\begin{aligned} (k_1 + H) \otimes_{\mathbb{K}} (k_2 + H) &= (1 + H) \otimes_{\mathbb{K}} (k_1 k_2 + H) \\ &= (1 + H) \otimes_{\mathbb{K}} (0 + H) \\ &= 0. \end{aligned}$$

In particular, we have that $\text{Ker } \mu = 0$, so \mathbb{K}/H is trivially zero product determined. \square

3. DIRECT SUMS OF ALGEBRAS

In order to state and prove our second main result, we require the following terminology. Given algebras (A, μ) and (B, λ) , we may endow their module direct sum $A \oplus B$ with an algebra structure. Define

$$\mu \boxplus \lambda : (A \oplus B) \otimes (A \oplus B) \longrightarrow A \oplus B$$

by

$$\mu \boxplus \lambda((a_1, b_1) \otimes (a_2, b_2)) = (\mu(a_1 \otimes a_2), \lambda(b_1 \otimes b_2)).$$

$\mu \boxplus \lambda$ is seen to be well-defined by Lemma 1.2 after noting that the function

$$\overline{\mu \boxplus \lambda}((a_1, b_1), (a_2, b_2)) = (\mu(a_1 \otimes a_2), \lambda(b_1 \otimes b_2))$$

is bilinear. In this way, $(A \oplus B, \mu \boxplus \lambda)$ is an algebra. This agrees with the usual meaning of the direct sum of two algebras using component-wise multiplication, since

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2) = \mu \boxplus \lambda((a_1, b_1) \otimes (a_2, b_2)).$$

The above example illustrates how the direct sum of algebras can be constructed completely in terms of linear maps on tensor products. Our primary purpose in this section is to show that the direct sum $(\bigoplus_{i \in I} A_i, \boxplus_{i \in I} \mu_i)$ of an arbitrary set of algebras is zero product determined if and only if each component algebra (A_i, μ_i) is zero product determined.

Lemma 3.1. *Let A_i ($i \in I$) be modules.*

$$\left(\bigoplus_{i \in I} A_i \right) \otimes \left(\bigoplus_{i \in I} A_i \right) \cong \bigoplus_{i, j \in I} A_i \otimes A_j$$

Proof. The lemma is proved after making two application of Proposition 2.1 in Chapter XVI of [11]. \square

We will rely on this isomorphism freely without mention in what follows. In particular, we will use the lemma to define $\boxplus_{i \in I} \mu_i$ and derive some simple properties.

Definition 3.2. Given algebras (A_i, μ_i) ($i \in I$), their *direct sum* as an algebra is the pair (A, μ) where $A = \bigoplus_{i \in I} A_i$, and

$$\mu = \boxplus_{i \in I} \mu_i : \bigoplus_{i, j \in I} A_i \otimes A_j \longrightarrow A$$

is defined by

$$\mu|_{A_i \otimes A_j} = \begin{cases} \mu_i & : i = j \\ 0 & : i \neq j \end{cases}$$

Since $\mu|_{A_i \otimes A_j}$ is a linear map on each component $A_i \otimes A_j$ of A , the map $\mu = \bigoplus_{i, j} \mu|_{A_i \otimes A_j}$ is a well-defined linear map on A by the universal property of module direct sums. In particular, we note the fact that

$$\mu(A_i \otimes A_i) \subset A_i.$$

Our second main result below shows that a direct sum of algebras is zero product determined if and only if each of its components is.

Theorem 3.3. *Let (A_i, μ_i) ($i \in I$) be algebras. Let (A, μ) be their direct sum as defined above. Then (A, μ) is zero product determined if and only if (A_i, μ_i) is zero product determined for all $i \in I$.*

To prove Theorem 3.3, we need the following two lemmas about (A, μ) .

Lemma 3.4. $\text{Ker } \mu = \left(\bigoplus_{i \in I} \text{Ker } \mu_i \right) \oplus \left(\bigoplus_{i, j \in I, i \neq j} A_i \otimes A_j \right).$

Proof. Clearly $\text{Ker } \mu \supset \left(\bigoplus_{i \in I} \text{Ker } \mu_i \right) \oplus \left(\bigoplus_{i, j \in I, i \neq j} A_i \otimes A_j \right)$. We will demonstrate the reverse inclusion. Denote by $p_{i,j}$ the canonical projection $p_{i,j} : A \longrightarrow A_i \otimes A_j$. Let $a \in \text{Ker } \mu$. Then

$$\begin{aligned} 0 &= \mu(a) \\ &= \sum_{i, j \in I} \mu(p_{i,j}(a)) \\ &= \sum_{i \in I} \mu(p_{i,i}(a)). \end{aligned}$$

Each $\mu(p_{i,i}(a)) \in A_i$, so each $\mu(p_{i,i}(a)) = 0$ by linear independence, giving $a \in \left(\bigoplus_{i \in I} \text{Ker } \mu_i \right) \oplus \left(\bigoplus_{i, j \in I, i \neq j} A_i \otimes A_j \right)$. \square

Lemma 3.5. $\langle T_\mu \rangle = \left(\bigoplus_{i \in I} \langle T_{\mu_i} \rangle \right) \oplus \left(\bigoplus_{i, j \in I, i \neq j} A_i \otimes A_j \right).$

Proof. By Lemma 3.4, $T_{\mu_i} \subset T_\mu$ for $i \in I$ and $A_i \otimes A_j \subset \langle T_\mu \rangle$ for $i, j \in I$ and $i \neq j$. Therefore,

$$\left(\bigoplus_{i \in I} \langle T_{\mu_i} \rangle \right) \oplus \left(\bigoplus_{i, j \in I, i \neq j} A_i \otimes A_j \right) \subset \langle T_\mu \rangle.$$

Conversely, every element of T_μ is of the form

$$a \otimes a' = \left(\sum_{i \in I} a_i \right) \otimes \left(\sum_{i \in I} a'_i \right) = \sum_{i,j \in I} a_i \otimes a'_j,$$

where $a_i, a'_i \in A_i$, $a_i = 0$ and $a'_i = 0$ for all but finitely many i , and

$$0 = \mu(a \otimes a') = \sum_{i,j \in I} \delta_{ij} \mu_i(a_i \otimes a'_j) = \sum_{i \in I} \mu_i(a_i \otimes a'_i).$$

Therefore, $a_i \otimes a'_i \in T_{\mu_i}$ and

$$a \otimes a' \in \left(\bigoplus_{i \in I} \langle T_{\mu_i} \rangle \right) \oplus \left(\bigoplus_{i,j \in I, i \neq j} A_i \otimes A_j \right).$$

This completes the proof. \square

Proof of Theorem 3.3. By Theorem 2.4, (A, μ) is zero product determined if and only if $\text{Ker } \mu = \langle T_\mu \rangle$. By Lemmas 3.4 and 3.5, $\text{Ker } \mu = \langle T_\mu \rangle$ if and only if

$$\left(\bigoplus_{i \in I} \text{Ker } \mu_i \right) \oplus \left(\bigoplus_{i,j \in I, i \neq j} A_i \otimes A_j \right) = \left(\bigoplus_{i \in I} \langle T_{\mu_i} \rangle \right) \oplus \left(\bigoplus_{i,j \in I, i \neq j} A_i \otimes A_j \right),$$

that is, if and only if $\text{Ker } \mu_i = \langle T_{\mu_i} \rangle$ for all $i \in I$, and in turn $\text{Ker } \mu_i = \langle T_{\mu_i} \rangle$ for all $i \in I$ if and only if (A_i, μ_i) is zero product determined for all $i \in I$. \square

4. APPLICATIONS TO LIE ALGEBRAS

An abelian Lie algebra A is a Lie algebra with trivial multiplication. In other words, $\text{Ker } \mu = A \otimes A$, so that A is trivially seen to be zero product determined.

Lemma 4.1. *Let A be an abelian Lie algebra. Then A is zero product determined.*

Proof. That A is abelian means that $T_\mu = T$, so that

$$\langle T_\mu \rangle = \langle T \rangle = A \otimes A = \text{Ker } \mu.$$

By Theorem 2.4, A is zero product determined. \square

Recall that a semi-simple Lie algebra L decomposes as the direct sum of simple ideals $L = L_1 \oplus \cdots \oplus L_r$ and that a reductive Lie algebra L decomposes as $L = L_0 \oplus L_1 \oplus \cdots \oplus L_r$, where L_0 is the center of L and $L_1 \oplus \cdots \oplus L_r$ is semi-simple. See [4] and [10] for details.

In [12], Wang, et. al. show that every parabolic subalgebra of a finite-dimensional simple Lie algebra over an algebraically-closed field of characteristic zero is zero product determined. Combining this result with our result for direct sums (Theorem 3.3) and the above lemma significantly broadens the class of Lie algebras known to be zero-product determined. Below is our third main result.

Theorem 4.2. *Let L be a reductive Lie algebra over an algebraically closed field \mathbb{K} of characteristic 0. Then every parabolic subalgebra P of L is zero product determined. In particular, L is zero product determined.*

Proof. The reductive Lie algebra L can be decomposed as $L = L_0 \oplus L_1 \oplus \cdots \oplus L_r$, where $L_0 = Z(L)$ is an abelian ideal, and L_1, \dots, L_r are simple ideals of L . Every parabolic subalgebra P of L is isomorphic by some automorphism to a standard parabolic subalgebra P' of the form $L_0 \oplus P_1 \oplus \cdots \oplus P_r$, where P_i is a parabolic subalgebra of L_i for $i = 1, \dots, r$. Then P' is zero product determined by Theorem 3.3, whence so is P . \square

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